Problem Set 1: Rings and Modules

Throughout the following exercises $A$ denotes a ring.

1. Let $\phi : A \to B$ be a homomorphism of rings. If $J$ is an ideal of $B$, then show that $\phi^{-1}(J)$ is an ideal of $A$. Further, show that $J \in \text{Spec} (B)$ implies $\phi^{-1}(J) \in \text{Spec} (A)$. Is it true that $J \in \text{Max} (B)$ implies $\phi^{-1}(J) \in \text{Max} (A)$? Also, is it true that if $I$ is an ideal of $A$, then $\phi(I)$ is an ideal of $B$? What if $\phi$ is surjective? Further, if $\phi$ is surjective, then is it true that $I \in \text{Spec} (A)$ implies $\phi(I) \in \text{Spec} (B)$, and that $I \in \text{Max} (A)$ implies $\phi(I) \in \text{Max} (B)$? Justify your answers.

2. Let $I$ be an ideal of $A$ and $q : A \to A/I$ be the natural homomorphism given by $x \mapsto x + I$. Show that $J \mapsto q(J)$ defines a bijective map from the ideals of $A$ containing $I$ and the ideals of $A/I$. Further, show that this bijection preserves inclusions, primality and maximality.

3. Assume that $A$ is a PID. Given any $a, b \in A$, let $d = \text{GCD}(a, b)$ and $\ell = \text{LCM}(a, b)$. If $I = (a)$ and $J = (b)$, then show that

$$IJ = (ab), \quad I \cap J = (\ell), \quad I + J = (d), \quad \text{and} \quad (I : J) = (a/d).$$

Are these results valid if $A$ is an arbitrary ring. What if $A$ is a UFD? Justify your answer.

4. Consider ideals $a, b, c$ of $A$ and the following three equalities.

$$ab = a \cap b, \quad (a + b)(a \cap b) = ab, \quad a \cap (b + c) = (a \cap b) + (a \cap c).$$

In each case, determine if the equality is valid for arbitrary $a, b, c$. If yes, then give a proof; otherwise give a counterexample. Also, if the answer is no, then determine if either of the inclusions $\subseteq$ and $\supseteq$ is valid, in general.

5. If $I$ is an ideal of $A$, then show that $\sqrt{I}$ is an ideal of $A$.

6. Show that colons commute with intersections, whereas radicals commute with finite intersections. More precisely, if $\{I_\alpha : \alpha \in \Lambda\}$ is a family of ideals of a ring $A$ and $J$ is any ideal of $A$, then show that

$$\bigcap_{\alpha \in \Lambda} (I_\alpha : J) = \left( \bigcap_{\alpha \in \Lambda} I_\alpha : J \right) \quad \text{and if } \Lambda \text{ is finite, then } \sqrt{\bigcap_{\alpha \in \Lambda} I_\alpha} = \bigcap_{\alpha \in \Lambda} \sqrt{I_\alpha}.$$ 

Give examples to show that these results do not hold (for finite families) if intersections are replaced by products.
7. Suppose $A$ is not the zero ring and let $\mathfrak{N}$ be the nilradical of $A$. Show that the following are equivalent.

(i) $A$ has exactly one prime ideal.
(ii) Every element of $A$ is either a unit or a nilpotent.
(iii) $A/\mathfrak{N}$ is a field.

8. Let $k$ be a field. Show that $\dim_k k[X_1, \ldots, X_n]_d = \binom{n+d-1}{d}$. 

9. Let $f(X) = a_0 + a_1 X + \ldots + a_n X^n \in A[X]$. Prove the following:

(i) $f$ is a unit in $A[X]$ if and only if $a_0$ is unit in $A$ and $a_1, a_2, \ldots, a_n$ are nilpotent in $A$.
(ii) $f$ is a nilpotent in $A[X]$ if and only if $a_0, a_1, \ldots, a_n$ are nilpotent in $A$.
(iii) $f$ is a zero divisor in $A[X]$ if and only if there exists $a \in A$ with $a \neq 0$ such that $af = 0$.

10. Let $S$ be any multiplicatively closed subset of $A$. Consider the relation on $A \times S$ defined by $(a, s) \sim (b, t) \iff (at - bs) = 0$. Determine if $\sim$ is an equivalence relation.

11. Given any $f \in A$, let $S = \{ f^n : n \in \mathbb{N} \}$ and $A_f = S^{-1}A$. Show that $A_f$ is isomorphic to $A[X]/(X f - 1)$.

12. Let $S$ and $T$ be multiplicatively closed subsets of $A$ with $S \subseteq T$ and let $U$ denote the image of $T$ under the natural map $\phi : A \to S^{-1}A$. Show that $T^{-1}A$ is isomorphic to $U^{-1}(S^{-1}A)$.

13. Show that localization commutes with taking homomorphic images. More precisely, if $I$ is an ideal of a ring $A$ and $S$ is a multiplicatively closed subset of $A$, then show that $S^{-1}A/S^{-1}I$ is isomorphic to $S^{-1}(A/I)$, where $\overline{S}$ denotes the image of $S$ in $A/I$.

14. Let $A$ be an integral domain. Fix a quotient field $K$ of $A$ and consider the localization $A_p$, where $p \in \text{Spec}(A)$, as subrings of $K$. Show that

$$A = \bigcap_{p \in \text{Spec}(A)} A_p = \bigcap_{m \in \text{Max}(A)} A_m.$$ 

15. Consider the following ring-theoretic properties that $A$ can have: (i) integral domain, (ii) field, (iii) PIR, (iv) PID, and (v) UFD. For each of these, determine if the property is preserved under the passage from $A$ to a (i) residue class ring, (ii) polynomial ring, or (iii) localization.
16. Let $M$ be an $A$-module and $S$ be a multiplicatively closed subset of $A$. Define carefully the localization $S^{-1}M$ of $M$ at $S$. With ideals replaced by $A$-submodules, determine which of the notions and results concerning localization of rings have an analogoue in the setting of modules.

17. Let $(A, m)$ be a local ring [which means that $A$ is a local ring and $m$ is its unique maximal ideal] and $M$ be a finitely generated $A$-module. For $x \in M$, let $\overline{x}$ denote the image of $x$ in the $A/m$-module $M/mM$. Given any $x_1, \ldots, x_r \in M$, show that $\{x_1, \ldots, x_r\}$ is a minimal set of generators of $M$ if and only if $\{\overline{x}_1, \ldots, \overline{x}_r\}$ is a basis for the $A/m$-vector space $M/mM$. Deduce that any two minimal set of generators of $M$ have the same cardinality, namely, $\dim_{A/m} M/mM$.

18. Assume that $A$ is not the zero ring and let $m, n \in \mathbb{N}$. Use Exercise 17 to show that $A^m$ and $A^n$ are isomorphic as $A$-modules iff $m = n$.

19. Given any $f, g \in A$, show that the principal open sets $D_f$ and $D_g$ of $\text{Spec } A$ satisfy the following.
   (i) $D_f = \emptyset \iff f$ is nilpotent,  
   (ii) $D_f = \text{Spec } (A) \iff f$ is a unit,  
   (iii) $D_f \cap D_g = D_{fg}$, and  
   (iv) $D_f = D_g \iff \sqrt{(f)} = \sqrt{(g)}$.

20. Given any $f \in A$, show that the principal open set $D_f$ is quasi-compact. Further show that an open subset of $\text{Spec } (A)$ is quasi-compact if and only if it is a finite union of principal open sets.
Problem Set 2: Noetherian Rings and Modules

Throughout the following exercises $A$ denotes a ring.

1. Let $A = k[X, Y, Z, \ldots]$ be the polynomial ring in infinitely many variables with coefficients in a field $k$. Prove that $A$ is not noetherian.

2. Let $q$ be an ideal of $A$ and $p = \sqrt{q}$. Show that if $A$ is noetherian, then $p^n \subseteq q$ for some $n \in \mathbb{N}$. Is this result valid if $A$ is not noetherian? Justify your answer.

3. Let $q$ be a nonunit ideal of $A$. Show that $q$ is primary if and only if every zerodivisor in $A/q$ is nilpotent.

4. Let $q$ be a $p$-primary ideal and $x$ be an element of $A$. Show that if $x \in q$, then $(q : x) = (1)$, whereas if $x \notin q$, then $(q : x)$ is $p$-primary, and in particular, $\sqrt{(q : x)} = p$. Further show that if $x \notin p$, then $(q : x) = q$.

5. Show that if $q$ is an ideal of $A$ such that $\sqrt{q} \in \text{Max} (A)$, then $q$ is primary.

6. Let $A = \mathbb{Z}[X]$ and consider the ideals $q = (4, X)$ and $m = (2, X)$. Show that $m$ is a maximal ideal of $A$ and $q$ is $m$-primary, but $q$ is not a power of $m$.

7. Let $A = k[X, Y]$, $I = (X^2, XY, Y^2)$, and $J = (X^2, Y) \cap (X, Y^2)$ be a primary decomposition of $I$. Is this an irredundant primary decomposition of $I$? Justify your answer.

8. Let $I$ be a radical ideal and $I = q_1 \cap \cdots \cap q_h$ be an irredundant primary decomposition of $I$, where $q_i$ is $p_i$-primary for $1 \leq i \leq h$. Show that $I = p_1 \cap \cdots \cap p_h$. Deduce that $I$ has no embedded component, and that $q_i = p_i$ for $1 \leq i \leq h$.

9. Given an ideal $I$ of $A$, define $Z(A/I) := \{x \in A : (I : x) \neq I\} \cup \{0\}$. Show that $Z(A/I)$ is the union of the associated primes of $I$, that is,

$$Z(A/I) = \bigcup_{p \in \text{Ass}(A/I)} p$$

and deduce that $Z(A) = \bigcup_{p \in \text{Ass}(A/(0))} p$,

where $Z(A)$ denotes the set of all zerodivisors of $A$.

10. Let $I$ be a nonunit ideal of $A$ and $\text{Ass}(A/I)$ be the set of associated primes of $I$ in $A$. Show that the minimal elements in $\text{Ass}(A/I)$ are precisely the minimal elements in the set $V(I) = \{p \in \text{Spec} A : p \supseteq I\}$ of primes containing $I$. 

4
11. Let $S$ be a multiplicative closed subset of $A$ and $q$ be a $p$-primary ideal of $A$. Show that if $S \cap p \neq \emptyset$, then $S^{-1}q = S^{-1}A$, whereas if $S \cap p = \emptyset$, then $S^{-1}q$ is $S^{-1}p$-primary and $S^{-1}q \cap A = q$. Deduce that if $I$ is any ideal of $A$ and $I = q_1 \cap \cdots \cap q_h$ is a primary decomposition of $I$ in $A$, then
\[ S^{-1}I = \bigcap_{p \cap S = \emptyset} S^{-1}q_i \quad \text{and} \quad S^{-1}I \cap A = \bigcap_{p \cap S = \emptyset} q_i. \]

12. Let $A = k[X, Y, Z]/(XY - Z^2)$ and write $x, y, z$ for the images of $X, Y, Z$ in $A$, respectively. Show that $p = (x, z)$ is a prime ideal of $A$, but $p^2 = (x^2, xz, z^2)$ is not primary. Further show that $x \notin p^2$, but $x \in p^{(2)}$.


(i) $p \in \text{Spec}(A) \implies p[X] \in \text{Spec}(A[X]).$

(ii) $q$ is $p$-primary $\implies q[X]$ is $p[X]$-primary.

(iii) $p$ is a minimal prime of $I$ $\implies p[X]$ is a minimal prime of $I[X]$.

(iv) $I = \cap_{i=1}^n q_i$, a primary decomposition of $I$

$\implies I[X] = \cap_{i=1}^n q_i[X]$ a primary decomposition of $I[X]$.

14. Let $\Delta$ be a simplicial complex with vertex set $V = \{1, 2, \ldots, n\}$, and let $F_1, F_2, \ldots, F_m$ be the facets (i.e., maximal faces) of $\Delta$. Let $I_\Delta$ be the ideal of $k[X_1, \ldots, X_n]$ generated by the squarefree monomials $X_{i_1} \cdots X_{i_r}$ for which $\{i_1, \ldots, i_r\} \notin \Delta$. Given any face $F$ of $\Delta$, let $P_F$ be the ideal of $k[X_1, \ldots, X_n]$ generated by the variables $X_{j_1}, \ldots, X_{j_s}$, where $\{j_1, \ldots, j_s\} = V \setminus F$. Prove that each $P_F$ is a prime ideal and $I_\Delta = P_{F_1} \cap \cdots \cap P_{F_m}$ is an irredundant primary decomposition of $I_\Delta$.

15. Let $J$ be a monomial ideal of $k[X_1, \ldots, X_n]$ and $u, v$ be relatively prime monomials in $k[X_1, \ldots, X_n]$. Show that $(J, uv) = (J, u) \cap (J, v)$. Also show that if $e_1, \ldots, e_n$ are positive integers, then $(X_1^{e_1}, \ldots, X_n^{e_n})$ is $(X_1, \ldots, X_n)$-primary. Use these facts to determine the associated primes and a primary decomposition of the ideal $(X^2Y^2Z, Y^2Z, YZ^3)$ of $k[X, Y, Z]$.

16. Consider $\mathbb{Q}/\mathbb{Z}$ as a $\mathbb{Z}$-module. Determine if it is a noetherian module?
Problem Set 3: Dimension, Height, and Integral Extensions

Throughout the following exercises $A$ denotes a ring.

1. Assume that $A$ is a noetherian ring. If $a \in A$ is a nonzerodivisor and $p$ is a minimal prime of $(a)$, then prove that $ht\ p = 1$.

2. Give an example of a minimal prime $p$ of a principal ideal of a noetherian ring such that $ht\ p = 0$.

3. Prove the converse of Krull’s Principal Ideal Theorem: If $A$ is noetherian and $p \in \text{Spec}(A)$ has height $r$, then there exists $a_1, \ldots, a_r \in p$ such that $p$ is a minimal prime of $(a_1, \ldots, a_r)$.

4. Give an example of a prime ideal $p$ of height 1 in a noetherian ring $A$ such that $p$ is not principal.

5. Prove that prime ideals in a noetherian ring satisfy the descending chain condition.

6. Determine the dimension of the ring $\mathbb{Z}[X]$ of polynomials in one variable with integer coefficients.

7. Suppose $A$ is noetherian and $I$ is any ideal of $A$. Show that $\dim A/I = \max\{\dim A/p : p \in \text{Ass}(A/I)\} = \max\{\dim A/p : p \in \text{Min}(A/I)\}$.

8. Let $\Delta$ be a simplicial complex with vertex set $V = \{1, 2, \ldots, n\}$ and $I_\Delta$ be the ideal of $k[X_1, \ldots, X_n]$ as defined in Q. 14 of Problem Set 2. Consider the residue class ring $R_\Delta := k[X_1, \ldots, X_n]/I_\Delta$. Show that $\dim R_\Delta = d + 1$, where $d$ is the (topological) dimension of $\Delta$. [Note: $R_\Delta$ is called the face ring or the Stanley-Reisner ring associated to $\Delta$.]

9. If a rational number satisfies a monic polynomial in $\mathbb{Z}[X]$, then show that it must be an integer. Deduce that $\mathbb{Z}$ is a normal domain. More generally, show that any UFD is a normal domain.

10. If $B/A$ is an integral extension of rings, then show that $B/J$ is integral over $A/J \cap A$ for every ideal $J$ of $A$. Further, if $S$ is a multiplicatively closed subset of $A$, then show that $S^{-1}B$ is an integral extension of $S^{-1}A$.

11. If $A$ is a normal domain and $S$ is a multiplicatively closed subset of $A$ such that $0 \notin S$, then show that $S^{-1}A$ is a normal domain.

12. Show that if $A$ is a domain, then $A$ is normal if and only if $A[X]$ is normal.
13. If $A$ is a normal domain, $K$ is its quotient field, and $x$ is an element of a field extension $L$ of $K$ such that $x$ is integral over $A$, then show that the minimal polynomial of $x$ over $K$ has its coefficients in $A$.

14. Consider the subring $A = \mathbb{Z}[\sqrt{5}]$ of $\mathbb{C}$ and let $\alpha = (1 + \sqrt{5})/2$. Show that $\alpha$ is in the quotient field of $A$ and $\alpha$ is integral over $A$, but $\alpha \notin A$.

15. Suppose $k$ is an infinite field and $f \in k[X_1, \ldots, X_n]$ is a nonzero polynomial. Prove that there exist some $a_1, \ldots, a_n \in k$ such that $f(a_1, \ldots, a_n) \neq 0$. Further show that if $n \geq 1$ and $f$ is a nonconstant homogeneous polynomial in $k[X_1, \ldots, X_n]$, then there are $c_2, \ldots, c_n \in k$ such that $f(1, c_2, \ldots, c_n) \neq 0$.

16. Prove the Tilting of Axes Lemma without the hypothesis that $k$ is an infinite field by proceeding as follows. Given a nonconstant polynomial $f \in k[X_1, \ldots, X_n]$, let $e$ an integer greater than any of the exponents of $X_1, \ldots, X_n$ appearing in $f$, and let $m_i = e^{i-1}$ for $2 \leq i \leq n$. Show that if $X'_i = X_i - X_1^{m_i}$ (instead of $X'_i = X_i - c_i X_1$), then $f = c X_1^{m_1} + g_1 X_1^{m_1} + \cdots + g_m$ for some $c \in k$ with $c \neq 0$ and $g_1, \ldots, g_m \in k[X'_2, \ldots, X'_n]$. Now argue as in the case of infinite $k$.

17. Let $A = k[X_1, \ldots, X_n]$. Prove that $\text{ht}(X_1, \ldots, X_r) = r$ for $1 \leq r \leq n$.

18. Let $k$ be a field and $B$ be a domain and a f.g. algebra over $k$. Prove that the Krull dimension of $B$ is equal to the transcendence degree of (the quotient field of) $B$ over $k$.

19. A refined version of Noether’s Normalization Lemma is as follows.

Let $B = k[x_1, \ldots, x_n]$ be a f.g. algebra over a field $k$ and $J_1 \subseteq \cdots \subseteq J_m$ be a chain of nonunit ideals of $B$. Then there exist $\theta_1, \ldots, \theta_d \in B$ and nonnegative integers $r_1 \leq \cdots \leq r_m$ satisfying the following.

(i) $\theta_1, \ldots, \theta_d$ are algebraically independent over $k$,

(ii) $B$ is integral over $A = k[\theta_1, \ldots, \theta_d]$; in particular, $B$ is a finite $A$-module,

(iii) $J_i \cap A = (\theta_1, \ldots, \theta_r)A$ for $1 \leq i \leq m$.

Assume this, and use it to show that if $B$ is a domain and a f.g. algebra over a field $k$, then for any prime ideal $P$ of $B$, we have $\dim B = \text{ht} P + \dim B/P$.

20. Consider $B = k[X, Y, Z]/(XY, XZ) = k[x, y, z]$ and $\mathfrak{p} = (y, z)$. Show that $\dim B = 2$, whereas $\text{ht} \mathfrak{p} = 0$ and $\dim B/\mathfrak{p} = 1$. Deduce that the last assertion in the previous problem is false if $B$ is not a domain.